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On the Interchange of the Variables in Certain Linear Differential Operators

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II. *On the Interchange of the Variables in Certain Linear Differential Operators.*

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Contents.

1. The operators to be considered, include or involve all those which have presented themselves as annihilators and generators in recent theories of functional differential invariants, reciprocants, cyclicants, &c. The general form of the binary operators, operators whose arguments are the derivatives of one dependent with regard to one independent variable, which I propose first to consider, is adopted in accordance with that used in two remarkable papers by Major MacMahon.* They are his operators in four elements. The analogous ternary operators to which I subsequently devote attention, are distinct from his operators of six elements. Their arguments are the partial derivatives of one of three variables, supposed connected by a single relation, with regard to the two others.

* "The Theory of a Multilinear Partial Differential Operator with applications to the Theories of Invariants and Reciprocants," 'London Math. Soc. Proc.,' vol. 18, 1887, pp. 61–88. "The Algebra of Multilinear Partial Differential Operators," 'London Math. Soc. Proc.,' vol. 19, pp. 112–128.

The only previous contribution, of which I am aware, to the subject of the reversion of MacMahon operators, is a paper by Professor L. J. Rogers,* in which he obtains the operator reciprocal to $\{\mu, \nu;\ 1, 1\}$, and alludes to the self reciprocal property of V which is expressed with more precision in (38) below.

I. *Binary Operators.*

2. Let x and y be two variables connected by any relation. Let y_r denote $\frac{1}{r!}\frac{d^r y}{dx^r}$, and x_r denote $\frac{1}{r!}\frac{d^r x}{dy^r}$.

Let ξ and η be corresponding increments of x and y, so that by Taylor's theorem

$$\eta = y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots \qquad (1)$$

and

$$\xi = x_1\eta + x_2\eta^2 + x_3\eta^3 + \ldots, \qquad (2)$$

the one expansion being a reversion of the other.

Let $Y_s^{(m)}$ denote the coefficient of ξ^s in the expansion of η^m, *i.e.*, of $(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m$ in ascending integral powers of ξ; and $X_s^{(m)}$ the coefficient of η^s in the expansion of ξ^m, *i.e.*, of $(x_1\eta + x_2\eta^2 + x_3\eta^3 + \ldots)^m$ in ascending integral powers of η. It is supposed that m is not fractional. It need not, however, be positive. Nor is it necessary to exclude the value zero, which, though somewhat special, will be seen to be of importance later.

Let n be a positive or negative integer or zero, and let μ and ν be any numerical quantities.

Denote

$$\frac{1}{m}\Sigma\left\{(\mu + \nu s)\, X_s^{(m)} \frac{d}{dx_{n+s}}\right\} \text{ by } \{\mu, \nu;\ m, n\}_x, \qquad (3)$$

and

$$\frac{1}{m}\Sigma\left\{(\mu + \nu s)\, Y_s^{(m)} \frac{d}{dy_{n+s}}\right\} \text{ by } \{\mu, \nu;\ m, n\}_y, \qquad (4)$$

the summations being, with regard to s, which assumes in turn all integral values not less than the greater of the two m and $-n+1$, so that, if $m + n > 1$, only symbols of differentiation with regard to all derivatives from y_{m+n} onwards may occur, while, if $m + n \overline{\gtrless} 1$, symbols of differentiation with regard to all derivatives may be present.

It is the operators $\{\mu, \nu;\ m, n\}_x$ and $\{\mu, \nu;\ m, n\}_y$ of which I propose to speak as MacMahon operators in x and y, respectively, dependent. It will be seen upon reference to the first paper referred to above that they are the results of substitution in Major MacMahon's operator $(\mu, \nu;\ m, n)$ for

* "Note on Conjugate Annihilators of Homogeneous and Isobaric Differential Equations," 'Messenger of Mathematics,' vol. 18, pp. 153–158.

$$a_0,\ a_1,\ a_2,\ a_3,\ \ldots$$

of

$$0,\ x_1,\ x_2,\ x_3,\ \ldots \text{ in the one case,}$$

and of

$$0,\ y_1,\ y_2,\ y_3,\ \ldots \text{ in the other.}$$

MACMAHON himself generally takes them as meaning

$$y_2,\ y_3,\ y_4,\ y_5\ \ldots\ ,$$

a fact which must not be forgotten in connecting his results with those to be here obtained.

The essential difference between the cases of $m + n \nless 1$ and $m + n < 1$ should be noticed at the outset. In the former case, the complete set of coefficients $X_s^{(m)}$ appears in the operator $\{\mu,\ \nu;\ m,\ n\}_x$. In the latter, one or more of those coefficients (a number of them equal to the excess of $-n + 1$ over m) is wanting at the beginning.

3. The aim in view is to express any MacMahon operator $\{\mu,\ \nu;\ m,\ n\}_x$ in x dependent as an operator or sum of operators of like form $\{\mu',\ \nu';\ m',\ n'\}_y$ in y dependent. We need the linear expressions in d/dy_1, d/dy_2, d/dy_3, ... which, when operating on any function of y_1, y_2, y_3, ... are equivalent to d/dx_1, d/dx_2, d/dx_3, ... operating on the equal function of x_1, x_2, x_3, ... The expressions in question I have obtained in the second* of a series of papers on Cyclicants, &c. The best form for present purposes is hardly there given to the conclusions. It will therefore result in a gain of clearness and no loss of brevity if in the present article the proof is given rather than the result quoted. The same remark will apply to Article 17 below.

We may look upon x_1, x_2, x_3, ... as a number of independent quantities, upon y_1, y_2, y_3, ... as determinate functions of these quantities, and upon ξ and η as two other quantities connected with one another and with x_1, x_2, x_3, ... by (1) or its equivalent (2).

Give x_r alone of all the quantites x_1, x_2, x_3, ... an infinitesimal variation. Keep η constant. In virtue of (2) or its equivalent (1) ξ will vary in consequence of the variation of x_r. Also, as some or all of y_1, y_2, y_3, ... are functions of x_r, some or all of those quantities will vary. Thus, from (2) we shall obtain

$$\delta\xi = \eta^r\,\delta x_r,$$

and from (1)

$$0 = \{y_1 + 2y_2\xi + 3y_3\xi^2 + \ldots\}\,\delta\xi + \left\{\frac{dy_1}{dx_r}\,\xi + \frac{dy_2}{dx_r}\,\xi^2 + \frac{dy_3}{dx_r}\,\xi^3 + \ldots\right\}\delta x_r = 0.$$

* "On the Linear Partial Differential Equations satisfied by Pure Ternary Reciprocants," 'London Math. Soc. Proc.,' vol. 18, 1887, pp. 142–164.

Accordingly it follows that

$$\frac{dy_1}{dx_r}\xi + \frac{dy_2}{dx_r}\xi^2 + \frac{dy_3}{dx_r}\xi^3 + \ldots = -\eta^r\{y_1 + 2y_2\xi + 3y_3\xi^2 + \ldots\}; \quad . \quad (5)$$

and consequently, this being true for all values of ξ, that if by aid of (1) this right hand member be expanded in ascending powers of ξ, the coefficients of ξ, ξ^2, ξ^3, ... are exactly the expressions for dy_1/dx_r, dy_2/dx_r, dy_3/dx_r, ...

Now

$$\frac{d}{dx_r} = \frac{dy_1}{dx_r}\cdot\frac{d}{dy_1} + \frac{dy_2}{dx_r}\cdot\frac{d}{dy_2} + \frac{dy_3}{dx_r}\cdot\frac{d}{dy_3} + \ldots,$$

and is therefore the result of replacing each power ξ^s of ξ on the left, and therefore on the right, of (5) by the corresponding symbol d/dy_s. It follows that the expression on the right of (5) may be taken as a symbolical representation of the equivalent operator to d/dx_r, *i.e.*, that

$$\frac{d}{dx_r} = -(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^r(y_1 + 2y_2\xi + 3y_3\xi^2 + \ldots)$$

$$= -\eta^r\frac{d\eta}{d\xi}, \quad \ldots \quad (6)$$

where the meaning of the symbolisation on the right is that η is to be replaced by its equivalent in terms of ξ from (1), that the differentiation with regard to ξ is partial, that the product on the right is to be expanded as a sum of multiples of powers of ξ, and that then each power ξ^s is to be replaced by the corresponding symbol of differentiation d/dy_s.

4. The proof of (6) is the same for all positive integral values (including unity) of r. Thus the means of transforming any differential operator whatever is obtained.

The rule according to which any linear operator is transformed may be very simply expressed.

Exactly companion to the symbolical notation ξ^s for d/dy_s in an operator linear in d/dy_1, d/dy_2, d/dy_3, ... is the notation η^s for d/dx_s in an operator linear in d/dx_1, d/dx_2, d/dx_3, ... Now, writing a linear operator

$$A\frac{d}{dx_\alpha} + B\frac{d}{dx_\beta} + C\frac{d}{dx_\gamma} + \ldots$$

in the symbolical form

$$A\eta^\alpha + B\eta^\beta + C\eta^\gamma + \ldots,$$

we learn by (6) that its equivalent in d/dy_1, d/dy_2, d/y_3, ... is obtained by multiplication by $-d\eta/d\xi$, expansion in terms of ξ by (1), and substitution for each power ξ^s in the expanded result, of the corresponding d/dy_s.

5. Now the symbolical form of any MacMahon operator for which $m + n \nless 1$ is very simple. By (4) that of,

$$\{\mu, \nu\,;\ m, n\}_y$$

is

$$\frac{1}{m}\Sigma_{s \nless m}(\mu + \nu s)\,\mathrm{Y}_s^{(m)}\,\xi^{n+s}$$

i.e.

$$\frac{\mu}{m}\xi^n(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m + \frac{\nu}{m}\xi^{n+1}\frac{d}{d\xi}(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m$$

i.e.

$$\frac{\mu}{m}\xi^n\eta^m + \nu\xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi}. \qquad (7)$$

Thus in particular

$$\{1, 0\,;\ m, n,\}_y = \frac{1}{m}\xi^n\eta^m\,, \qquad (8)$$

and

$$\{0, 1\,;\ m, n\}_y = \xi^{n+1}\eta^{m-1}\frac{d\eta}{d\xi}\,, \qquad (9)$$

the right hand members being supposed to be expanded in terms of ξ by aid of (1), and then to have each power ξ^s of ξ which occurs replaced by the corresponding d/dy_s.

We may of course write (7)

$$\{\mu, \nu\,;\ m, n\}_y = \mu\{1, 0\,;\ m, n\}_y + \nu\{0, 1\,;\ m, n\}_y, \qquad (10)$$

so that in (8) and (9) we have involved all MacMahon operators in y for which $m + n$ is not less than unity. A reservation must for the moment be made of the case $m = 0$.

Exactly corresponding to (8) and (9) we have the symbolical forms

$$\{1, 0\,;\ m, n\}_x = \frac{1}{m}\eta^n\xi^m\,, \qquad (11)$$

$$\{0, 1\,;\ m, n\}_x = \eta^{n+1}\xi^{m-1}\frac{d\xi}{d\eta}\,, \qquad (12)$$

where the expansions on the right are to be in ascending powers of η by (2), and where in an expansion each η^s is to be replaced by the corresponding d/dx_s.

6. The transformation of $\{\mu, \nu\,;\ m, n\}_x$ for the cases at present under consideration of $m + n$ not less than unity is now immediate. By Art. 4 the transformed form of the expansion in terms of η of

$$\eta^n\xi^m\,,$$

considered as the symbolical form of an operator in x dependent, is the expansion in terms of ξ of

$$-\xi^m \eta^n \frac{d\eta}{d\xi},$$

considered as an operator in y dependent.

In other words, by (11) and (9),

$$m\{1, 0;\ m, n\}_x = -\{0, 1;\ n+1, m-1\}_y \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Again, the transformed form of the symbolical expansion in powers of η of

$$\eta^{n+1}\xi^{m-1}\frac{d\xi}{d\eta}$$

is, by Art. 4, the symbolical expansion in powers of ξ of

$$-\xi^{m-1}\eta^{n+1}\frac{d\xi}{d\eta}\cdot\frac{d\eta}{d\xi},$$

i.e., of

$$-\xi^{m-1}\eta^{n+1},$$

since in $d\xi/d\eta$ and $d\eta/d\xi$ the derivatives $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are not regarded as variables. In other words, by (12) and (8),

$$\{0, 1;\ m, n\}_x = -(n+1)\{1, 0;\ n+1, m-1\}_y \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It is to be remarked that (13) and (14) are entirely in accord. Either of them is produced from the other by the interchange of x and y and of m and $n+1$.

From (13) and (14) by aid of (10) we produce the more general equality of operators

$$\{\mu, \nu;\ m, n\}_x = -\left\{\nu(n+1), \frac{\mu}{m};\ n+1, m-1\right\}_y \quad . \quad . \quad . \quad . \quad (15)$$

which may be given the more symmetrical form

$$\{m\mu, \mu';\ m, m'-1\}_x = -\{m'\mu', \mu;\ m', m-1\}_y, \quad . \quad . \quad . \quad . \quad (16)$$

in which $m+m'$ has to be positive.

In (16) are included two interesting classes of particular cases, viz.:—

$$\{-m, 1;\ m, m-1\}_x = \{-m, 1;\ m, m-1\}_y, \quad . \quad . \quad . \quad . \quad (17)$$

and

$$\{m, 1;\ m, m-1\}_x = -\{m, 1;\ m, m-1\}_y \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Corresponding to each positive degree m there are then two self-reciprocal operators.* The first is of positive character, being entirely unaltered in form by

* Self reciprocal operators, of course, generate from absolute reciprocants other absolute reciprocants.

interchange of x and y; and the second of negative character, persisting in form but for a change of sign. (A complex self reciprocal operator can of course be found by taking the sum or difference of any two correlative operators; *e.g.*,

$$m\{1, 0;\ m, n\}_x \mp \{0, 1;\ n+1, m-1\}_x).$$

At greater length (17) and (18) are

$$\mathrm{X}^{(m)}_{m+1}\frac{d}{dx_{2m}} + 2\mathrm{X}^{(m)}_{m+2}\frac{d}{dx_{2m+1}} + 3\mathrm{X}^{(m)}_{m+3}\frac{d}{dx_{2m+2}} + \dots$$

$$= \mathrm{Y}^{(m)}_{m+1}\frac{d}{dy_{2m}} + 2\mathrm{Y}^{(m)}_{m+2}\frac{d}{dy_{m+1}} + 3\mathrm{Y}^{(m)}_{m+3}\frac{d}{dy_{m+2}} + \dots, \quad . \quad . \quad . \quad . \quad (17\mathrm{A})$$

and

$$2m\,\mathrm{X}^{(m)}_{m}\frac{d}{dx_{2m-1}} + (2m+1)\,\mathrm{X}^{(m)}_{m+1}\frac{d}{dx_{2m}} + (2m+2)\,\mathrm{X}^{(m)}_{m+2}\frac{d}{dx_{2m+1}} + \dots$$

$$= -\left\{2m\,\mathrm{Y}^{(m)}_{m}\frac{d}{dy_{2m-1}} + (2m+1)\,\mathrm{Y}^{(m)}_{m+1}\frac{d}{dy_{2m}} + (2m+2)\,\mathrm{Y}^{(m)}_{m+2}\frac{d}{dy_{2m+1}} + \dots\right\} \quad (18\mathrm{A})$$

In particular for $m = 1$ we have

$$x_2\frac{d}{dx_2} + 2x_3\frac{d}{dx_3} + 3x_4\frac{d}{dx_4} + \dots = y_2\frac{d}{dy_2} + 2y_3\frac{d}{dy_3} + 3y_4\frac{d}{dy_4} + \dots \quad (19)$$

and

$$2x_1\frac{d}{dx_1} + 3x_2\frac{d}{dx_2} + 4x_3\frac{d}{dx_3} + \dots = -\left\{2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + 4y_3\frac{d}{dy_3} + \dots\right\} \quad (20)$$

Again $m = 2$ gives us that

$$2x_1x_2\frac{d}{dx_4} + 2\,(2x_1x_3 + x_2^2)\frac{d}{dx_5} + 3\,(2x_1x_4 + 2x_2x_3)\frac{d}{dx_6}$$

$$+ 4\,(2x_1x_5 + 2x_2x_4 + x_3^2)\frac{d}{dx_7} + \dots, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and

$$4x_1^2\frac{d}{dx_3} + 5\,.\,2x_1x_2\frac{d}{dx_4} + 6\,(2x_1x_3 + x_2^2)\frac{d}{dx_5} + 7\,(2x_1x_4 + 2x_2x_3)\frac{d}{dx_6} + \dots, \quad (22)$$

are self reciprocal operators of positive and negative characters respectively.

7. As other examples of the important formulæ of transformation (13) and (14), let us write down cases corresponding to $m = 1$, $n \nless 0$.

For $m = 1$, $n = 0$ we obtain

$$x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2} + x_3\frac{d}{dx_3} + \dots = -\left\{y_1\frac{d}{dy_1} + 2y_2\frac{d}{dy_2} + 3y_3\frac{d}{dy_3} + \dots\right\}, \quad (23)$$

$$x_1\frac{d}{dx_1} + 2x_2\frac{d}{dx_2} + 3x_3\frac{d}{dx_3} + \dots = -\left\{y_1\frac{d}{dy_1} + y_2\frac{d}{dy_2} + y_3\frac{d}{dy_3} + \dots\right\}, \quad (24)$$

which together are equivalent to (19) and (20) together. From (23) it follows that a homogeneous function of $x_1\, x_2\, x_3, \ldots$ transforms into an isobaric function of $y_1\, y_2\, y_3, \ldots$, and that, i and w meaning degree and weight respectively,

$$i_x = -w_y,$$

while from (24) follows the equivalent fact that an isobaric function transforms into a homogeneous one, and that

$$w_x = -i_y.$$

From (19) follows the especially interesting fact that, if a function of $x_1, x_2\, x_3, \ldots$ is isobaric in $x_2, x_3, x_4, \ldots$ upon considering the weight of x_r to be $r-1$, so also is the transformed function of $y_1, y_2, y_3, \ldots$ isobaric in the same sense and of the same weight in $y_2, y_3, y_4, \ldots$

Again the substitution $m = 1$, $n = 1$ in (13) and (14) produces for us

$$x_1 \frac{d}{dx_2} + x_2 \frac{d}{dx_3} + x_3 \frac{d}{dx_4} + \ldots = -\tfrac{1}{2}\left\{2\mathrm{Y}_2^{(2)} \frac{d}{dy_2} + 3\mathrm{Y}_3^{(2)} \frac{d}{dy_3} + 4\mathrm{Y}_4^{(2)} \frac{d}{dy_4} + \ldots\right\}, \tag{25}$$

and

$$x_1 \frac{d}{dx_2} + 2x_2 \frac{d}{dx_3} + 3x_3 \frac{d}{dx_4} + \ldots = -\left\{\mathrm{Y}_2^{(2)} \frac{d}{dy_2} + \mathrm{Y}_3^{(2)} \frac{d}{dy_3} + \mathrm{Y}_4^{(2)} \frac{d}{dy_4} + \ldots\right\}, \tag{26}$$

where

$$\mathrm{Y}_2^{(2)} = y_1^2,\ \mathrm{Y}_3^{(2)} = 2y_1y_2,\ \mathrm{Y}_4^{(2)} = 2y_1y_3 + y_2^2,\ \mathrm{Y}_5^{(2)} = 2y_1y_4 + 2y_2y_3, \ldots$$

These two transformations have been obtained by Professor ROGERS (see note to Art. 1). The second tells us that what he calls primary invariants in $x_1, x_2, x_3, \ldots$ have for their transforms what he calls secondary invariants in $y_1, y_2, y_3, \ldots$

We might now consider the results of putting $m = 1$ and $n = 2, 3, \ldots$ in (13) and (14). By this means the transformation of lineo-linear operators of two, three, &c., steps is effected. For the general case $m = 1$, $n = n$ the results are

$$\begin{aligned} x_1 \frac{d}{dx_{n+1}} &+ x_2 \frac{d}{dx_{n+2}} + x_3 \frac{d}{dx_{n+3}} + \ldots \\ &= -\frac{1}{n+1}\left\{(n+1)\,\mathrm{Y}_{(n+1)}^{(n+1)} \frac{d}{dy_{n+1}} + (n+2)\,\mathrm{Y}_{(n+2)}^{(n+1)} \frac{d}{dy_{n+2}} + \ldots\right\} . \end{aligned} \tag{27}$$

$$x_1 \frac{d}{dx_{n+1}} + 2x_2 \frac{d}{dx_{n+2}} + 3x_3 \frac{d}{dx_{n+3}} + \ldots = -\left\{\mathrm{Y}_{(n+1)}^{(n+1)} \frac{d}{dy_{n+1}} + \mathrm{Y}_{(n+2)}^{(n+1)} \frac{d}{dy_{n+2}} + \ldots\right\} \tag{28}$$

Perhaps the most interesting fact to be deduced from (25) and (26) is the transformation of $\{-1, 1; 1, 1\}$, the second annihilator Ω of projective reciprocants. By subtraction of (25) from (26), or directly from (15)

$$\{-1, 1; 1, 1\}_x = \{-2, 1; 2, 0\}_y,$$

i.e.,

$$x_2 \frac{d}{dx_3} + 2x_3 \frac{d}{dx_4} + 3x_4 \frac{d}{dx_5} + \ldots$$

$$= \tfrac{1}{2} \left\{ Y_3^{(2)} \frac{d}{dy_3} + 2Y_4^{(2)} \frac{d}{dy_4} + 3Y_5^{(2)} \frac{d}{dy_5} + \ldots \right\}$$

$$= \tfrac{1}{2} \left\{ 2y_1 y_2 \frac{d}{dy_3} + 2(2y_1 y_3 + y_2^2) \frac{d}{dy_4} + 3(2y_1 y_4 + 2y_2 y_3) \frac{d}{dy_5} + \ldots \right\}$$

$$= y_1 \left\{ y_2 \frac{d}{dy_3} + 2y_3 \frac{d}{dy_4} + 3y_4 \frac{d}{dy_5} + \ldots \right\} + 2 \frac{y_2^2}{2} \frac{d}{dy_4} + 3y_2 y_3 \frac{d}{dy_5} + 4\left(y_2 y_4 + \frac{y_3^2}{2}\right) \frac{d}{dy_6} + \ldots,$$

or

$$\Omega(x, y) = y_1 \Omega(y, x) + 2 \frac{y_2^2}{2} \frac{d}{dy_4} + 3y_2 y_3 \frac{d}{dy_5} + 4\left(y_2 y_4 + \frac{y_3^2}{2}\right) \frac{d}{dy_6} + \ldots \qquad (29)$$

Since $y_1 x_1 = 1$ we infer from this conclusion and its correlative that

$$x_1^{-\frac{1}{2}} \left\{ 2 \frac{x_2^2}{2} \frac{d}{dx_4} + 3x_2 x_3 \frac{d}{dx_5} + 4\left(x_2 x_4 + \frac{x_3^2}{2}\right) \frac{d}{dx_6} + \ldots \right\} = y_1^{\frac{1}{2}} \Omega(y, x) - x_1^{\frac{1}{2}} \Omega(x, y)$$

$$= - y_1^{-\frac{1}{2}} \left\{ 2 \frac{y_2^2}{2} \frac{d}{dy_3} + 3y_2 y_3 \frac{d}{dy_4} + 4\left(y_2 y_4 + \frac{y_3^2}{2}\right) \frac{d}{dy_5} + \ldots \right\} \quad . \quad (30)$$

is a self reciprocal operator of negative character. The operator is one of considerable interest in connection with the theories of invariants and reciprocants. (See MacMahon, 'London Math. Soc. Proc.,' vol. 18, p. 75).

It also follows that the sum of $2x_1^{\frac{1}{2}} \Omega(x, y)$ and the operator on the left of (30) is a self reciprocal operator of positive character.

8. To complete our theory of the reversion of MacMahon operators, for which $m + n$ is not less than unity, we must consider the somewhat special and exactly correlative cases $m = 0$, $n \nless 1$, and $n = -1$, $m \nless 2$.

The operator $0\{1, 0; 0, n\}_x$ is d/dx_n in accordance with the general definition of $m\{\mu, \nu; m, n\}_x$ in (3). Thus the identity (6) may be written, by aid of (9),

$$0\{1, 0; 0, r\}_x = -\{0, 1; r+1, -1\}_y, \quad . \quad . \quad . \quad . \quad . \quad (31)$$

for any positive integral value, not excluding unity, of r: which is strictly in agreement with the general formula of transformation (13).

On the other hand the general definition (3) gives to $0\{0, 1; 0, n\}_x$ no other meaning than zero. So far then the operator $\{0, 1; 0, n\}_x$ is indeterminate in form. An interpretation of it is now sought which shall make the case not exceptional to the general formula of transformation (14).

To discover this interpretation let us reverse the order of investigation and seek the operator in x, which is equivalent to the operator in y obtained by putting $m = 0$ in the right hand member of (14).

By (8) the symbolical form of

$$-(n+1)\{1, 0 ; n+1, -1\}_y$$

is

$$-\xi^{-1}\eta^{n+1}.$$

The equivalent operator in x dependent has then for its symbolical form, as in Article 4,

$$\eta^{n+1}\xi^{-1}\frac{d\xi}{d\eta},$$

i.e.,

$$\eta^n + \eta^{n+1}\frac{d}{d\eta}\log\frac{\xi}{\eta}.$$

Now

$$\frac{d}{d\eta}\log\frac{\xi}{\eta} = \frac{d}{d\eta}\log(x_1 + x_2\eta + x_3\eta^2 + \ldots)$$

$$= \frac{d}{d\eta}\, e^{\eta(x_2 d/dx_1 + 2x_3 d/dx_2 + 3x_4 d/dx_3 + \ldots)}\cdot\log x_1.$$

$$= e^{\eta(x_2 d/dx_1 + 2x_3 d/dx_2 + 3x_4 d/dx_3 + \ldots)}\cdot\frac{x_2}{x_1},$$

$$= \frac{x_2}{x_1} + \frac{2x_1x_3 - x_2^2}{x_1^2}\eta + \frac{3x_1^2x_4 - 3x_1x_2x_3 + x_2^3}{x_1^3}\eta^2 + \ldots$$

$$= \frac{x_2}{x_1} + \frac{\mathrm{G}x_2}{x_1^2}\eta + \frac{\mathrm{G}^2x_2}{x_1^3}\cdot\frac{\eta^2}{1.2} + \frac{\mathrm{G}^3x_2}{x_1^4}\cdot\frac{\eta^3}{1.2.3} + \ldots, \quad . \quad . \quad . \quad . \quad (32)$$

where $\mathrm{G} = x_1\left(x_2\frac{d}{dx_1} + 2x_3\frac{d}{dx_2} + 3x_4\frac{d}{dx_3} + \ldots\right) - x_2\left(x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2} + x_3\frac{d}{dx_3} + \ldots\right)$

$$= (2x_1x_3 - x_2^2)\frac{d}{dx_2} + (3x_1x_4 - x_2x_3)\frac{d}{dx_3} + (4x_1x_5 - x_2x_4)\frac{d}{dx_4} + \ldots, \quad (32\text{A})$$

so that the numerators $\mathrm{G}x_2$, G^2x_2, G^3x_2, ..., are a set of seminvariant protomorphs in x_1, x_2, $2!\ x_3$, $3!\ x_4$, ...

Consequently the transformation in x dependent of $-(n+1)\{1, 0; n+1, -1\}_y$ is

$$\frac{d}{dx_n} + \frac{x_2}{x_1}\frac{d}{dx_{n+1}} + \frac{\mathrm{G}x_2}{x_1^2}\frac{d}{dx_{n+2}} + \frac{\mathrm{G}^2x_2}{2!\ x_1^3}\frac{d}{dx_{n+3}} + \frac{\mathrm{G}^3x_2}{3!\ x_1^4}\frac{d}{dx_{n+4}} + \ldots;$$

and it is accordingly this operator which has to be defined as

$$\{0, 1 ; 0, n\}_x^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

* *Cf.* HAMMOND, 'London Math. Soc. Proc.,' vol. 18, p. 64, note.

that (14) may be regarded as holding for the value $m = 0$ as well as for non-vanishing values of m.

It affords an instructive verification to conduct the investigation of the same transformation in the order of Article 6.

9. For the case $n = 1$ the two formulæ of transformation,

$$0\,\{1, 0;\ 0, n\}_x = -\{0, 1;\ n+1, -1\}_y,$$
$$\{0, 1;\ 0, n\}_x = -(n+1)\,\{1, 0;\ n+1, -1\}_y,$$

of the last article become respectively

$$\frac{d}{dx_1} = -\tfrac{1}{2}\left\{2Y_2^{(2)}\frac{d}{dy_1} + 3Y_3^{(2)}\frac{d}{dy_2} + 4Y_4^{(2)}\frac{d}{dy_3} + \dots\right\}$$
$$= -\tfrac{1}{2}\left\{2y_1^2\frac{d}{dy_1} + 3.2y_1y_2\frac{d}{dy_2} + 4\,(2y_1y_3 + y_2^2)\frac{d}{dy_3} + \dots\right\}, \quad . \quad (34)$$

and

$$\frac{d}{dx_1} + \frac{x_2}{x_1}\frac{d}{dx_2} + \frac{2x_1x_3 - x_2^2}{x_1^2}\frac{d}{dx_3} + \frac{3x_1^2x_4 - 3x_1x_2x_3 + x_2^3}{x_1^3}\frac{d}{dx_4} + \dots$$
$$= -\left\{y_1^2\frac{d}{dy_1} + 2y_1y_2\frac{d}{dy_2} + (2y_1y_3 + y_2^2)\frac{d}{dy_3} + \dots\right\} \quad . \quad . \quad . \quad . \quad (35)$$

By combination of these we have the equivalence, free from d/dx_1 and d/dy_1,

$$\frac{x_2}{x_1}\frac{d}{dx_2} + \frac{2x_1x_3 - x_2^2}{x_1^2}\frac{d}{dx_3} + \frac{3x_1^2x_4 - 3x_1x_2x_3 + x_2^3}{x_1^3}\frac{d}{dx_4} + \dots$$
$$= y_1y_2\frac{d}{dy_2} + 2\left(y_1y_3 + \frac{y_2^2}{2}\right)\frac{d}{dy_3} + 3\,(y_1y_4 + y_2y_3)\frac{d}{dy_4} + \dots$$
$$= y_1\left(y_2\frac{d}{dy_2} + 2y_3\frac{d}{dy_3} + 3y_4\frac{d}{dy_4} + \dots\right) + 2\frac{y_2^2}{2}\frac{d}{dy_3} + 3y_2y_3\frac{d}{dy_4}$$
$$+ 4\left(y_2y_4 + \frac{y_3^2}{2}\right)\frac{d}{dy_5} + \dots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

In like manner, for any positive integral value of n,

$$\{0, 1;\ 0, n\}_x - 0\{1, 0;\ 0, n\}_x = -(n+1)\,\{1, 0;\ n+1, -1\}_y$$
$$- \{0, 1;\ n+1, -1\}_y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

is an equivalence of operators which do not involve any lower symbols of operation than d/dx_{n+1} and d/dy_{n+1} respectively.

10. From (34) is easily derived in exact form the known fact that the annihilator V

of pure reciprocants is, when affected with a simple multiplier, self reciprocal. We may write (34)

$$\frac{d}{dx_1} = y_1^2\frac{d}{dy_1} - y_1\left\{2y_1\frac{d}{dy_1} + 3y_2\frac{d}{dy_2} + 4y_3\frac{d}{dy_3} + \ldots\right\}$$

$$- \left\{4\frac{y_2^2}{2}\frac{d}{dy_3} + 5y_2y_3\frac{d}{dy_4} + 6\left(y_2y_4 + \frac{y_3^2}{2}\right)\frac{d}{dy_5} + \ldots\right\}$$

$$= y_1^2\frac{d}{dy_1} - y_1\{1, 1;\ 1, 0\}_y - \text{V}(y, x)\,.$$

$$\therefore\ x_1\frac{d}{dx_1} - y_1\frac{d}{dy_1} = -\{1, 1;\ 1, 0\}_y - y_1^{-1}\text{V}(y, x)\,.$$

So too, correlatively,

$$y_1\frac{d}{dy_1} - x_1\frac{d}{dx_1} = -\{1, 1;\ 1, 0\}_x - x_1^{-1}\text{V}(x, y)\,.$$

But

$$\{1, 1;\ 1, 0\}_x = -\{1, 1;\ 1, 0\}_y \text{ by (18) or (20)}$$

$$\therefore\ x_1^{-1}\text{V}(x, y) = -y_1^{-1}\text{V}(y, x)\,; \qquad (38)$$

so that, to use a familiar notation, $t^{-1}\text{V}$ is a self reciprocal operator of negative character.

11. It remains to consider operators $\{\mu, \nu;\ m, n\}$ in cases when $m + n < 1$. In such cases the formulæ of Arts. 5 and 6 have to be replaced by others. The essential difference between them and the cases already considered lies in the fact that the lower limit of s in (3) and (4), and, therefore, in what replaces (7), is now $-n+1$ instead of m, *i.e.*, is greater than m, so that the coefficients in $\{\mu, \nu;\ m, n\}_y$ are no longer multiples of the complete set of coefficients in the expansion of $(y_1\xi + y_2\xi^2 + y_3\xi^3 + \ldots)^m$, but of those coefficients with the exception of one or more at the beginning.

In the present article attention is confined to the case of $m + n = 0$, *i.e.*, $n = -m$.

Proceeding to write down the symbolical form of $\{\mu, \nu;\ m, -m\}_y$ as in Art. (5) we see that the whole expansion from which we there started is present except the first term,

$$\frac{1}{m}(\mu + \nu m)\,\text{Y}_m^{(m)}\xi^{n+m}.$$

Thus the symbolical form of

$$\{1, 0;\ m, -m\}_y \ \text{ is } \ \frac{1}{m}\xi^{-m}(\eta^m - y_1^m\xi^m), \qquad (39)$$

and that of

$$\{0, 1;\ m, -m\}_y \ \text{ is } \ \xi^{-m+1}\left(\eta^{m-1}\frac{d\eta}{d\xi} - y_1^m\xi^{m-1}\right), \qquad (40)$$

the right-hand members standing for their expansions in powers of ξ.

In like manner

$$\frac{1}{m}\eta^{-m}(\xi^m - x_1{}^m\eta^m) \quad \text{and} \quad \eta^{-m+1}\left(\xi^{m-1}\frac{d\xi}{d\eta} - x_1{}^m\eta^{m-1}\right), \qquad \ldots (41, 42)$$

standing for their expansions in terms of η, are the symbolical forms of

$$\{1, 0; m, -m\}_x \quad \text{and} \quad \{0, 1; m, -m\}_x.$$

As in Art. (6) the result of transforming $\{1, 0; m, -m\}_x$ to its form in y dependent is, then, symbolically,

$$-\frac{1}{m}\eta^{-m}(\xi^m - x_1{}^m\eta^m)\frac{d\eta}{d\xi},$$

i.e.,

$$-\frac{1}{m}\xi^m\eta^{-m}\frac{d\eta}{d\xi} + \frac{1}{m}y_1{}^{-m}\frac{d\eta}{d\xi},$$

i.e.,

$$-\frac{1}{m}\xi^m\left(\eta^{-m}\frac{d\eta}{d\xi} - y_1{}^{-m+1}\xi^{-m}\right) + \frac{1}{m}y_1{}^{-m}\left(\frac{d\eta}{d\xi} - y_1\right);$$

whence, by aid of (40),

$$m\{1, 0; m, -m\}_x = -\{0, 1; 1-m, m-1\}_y + y_1{}^{-m}\{0, 1; 1, -1\}_y. \quad (43)$$

Once more the result of transforming $\{0, 1; m, -m\}_x$ is, in like manner,

$$-\eta^{-m+1}\left(\xi^{m-1}\frac{d\xi}{d\eta} - x_1{}^m\eta^{m-1}\right)\frac{d\eta}{d\xi},$$

i.e.,

$$-\xi^{m-1}\eta^{-m+1} + y_1{}^{-m}\frac{d\eta}{d\xi},$$

i.e.,

$$-\xi^{m-1}(\eta^{-m+1} - y_1{}^{-m+1}\xi^{-m+1}) + y_1{}^{-m}\left(\frac{d\eta}{d\xi} - y_1\right);$$

so that, by (39) and (40),

$$\{0, 1; m, -m\}_x = -(1-m)\{1, 0; 1-m, m-1\}_y + y_1{}^{-m}\{0\ 1; 1, -1\}_y. \quad (44)$$

From (43) and (44) follows the more general identity,

$$m\{\mu, \nu; m, -m\}_x = -\{\nu m(1-m), \mu; 1-m, m-1\}_y$$
$$+ (\mu + m\nu)y_1{}^{-m}\{0, 1; 1, -1\}_y;$$

or, replacing νm by ν,

$$m\left\{\mu, \frac{\nu}{m}; m, -m\right\}_x = -(1-m)\left\{\nu, \frac{\mu}{1-m}; 1-m, m-1\right\}_y + (\mu+\nu)\, y_1^{-m}\{0, 1; 1, -1\}_y \quad . \quad (45)$$

In particular,

$$m\left\{\mu, -\frac{\mu}{m}; m, -m\right\}_x = (1-m)\left\{\mu, -\frac{\mu}{1-m}; 1-m, m-1\right\}_y . \quad . \quad (46)$$

12. The value zero of m is somewhat special in these cases of $n = -m$, just as in the more general cases already discussed. So too, of course, is the conjugate value $m = 1$.

For (43) and (44) to hold for these special values of m we must have,

$$\left.\begin{aligned} 0.\{1, 0; 0, 0\}_x &= -\ \{0, 1; 1, -1\}_y + \{0, 1; 1, -1\}_y, \\ \{1, 0; 1, -1\}_x &= -\ \{0, 1; 0, 0\}_y \quad + y_1^{-1}\{0, 1; 1, -1\}_y, \\ \{0, 1; 0, 0\}_x &= -\ \{1, 0; 1, -1\}_y + \{0, 1; 1, -1\}_y, \\ \{0, 1; 1, -1\}_x &= -\ 0\,\{1, 0; 0, 0\}_y \quad + y_1^{-1}\{0, 1; 1, -1\}_y. \end{aligned}\right\} \quad . \quad (47)$$

Of these four equalities the first is a mere identity of two zero operators. In fact to $0\,\{1, 0; 0, 0\}$ no other meaning than zero could be attached consistently with the general definition. Thus the form of $\{1, 0; 0, 0\}$ is left indeterminate.

The fourth of (47) becomes

$$\{0, 1; 1, -1\}_x = y_1^{-1}\{0, 1; 1, -1\}_y,$$

i.e.,

$$x_1^{-\frac{1}{2}}\left\{2x_2\frac{d}{dx_1} + 3x_3\frac{d}{dx_2} + 4x_4\frac{d}{dx_3} + \ldots\right\} = y_1^{-\frac{1}{2}}\left\{2y_2\frac{d}{dy_1} + 3y_3\frac{d}{dy_2} + 4y_4\frac{d}{dy_3} + \ldots\right\}, \quad . \quad (48)$$

of which the left-hand member is merely $x_1^{-\frac{3}{2}}\frac{d}{dy}$, and the right $y_1^{-\frac{3}{2}}\frac{d}{dx}$, the symbols of differentiation being total.

The remaining equalities, the second and third of (47), now become the same but for an interchange of x and y. Consequently there will be complete consistency if we define the at present undetermined operator $\{0, 1; 0, 0\}$ as that which obeys the equation of transformation

$$\begin{aligned} \{0, 1; 0, 0\}_x &= -\{1, 0; 1, -1\}_y + \{0, 1; 1, -1\}_y \\ &= \{-1, 1; 1, -1\}_y \\ &= y_2\frac{d}{dy_1} + 2y_3\frac{d}{dy_2} + 3y_4\frac{d}{dy_3} + \ldots \quad . \quad (49) \end{aligned}$$

Now, proceeding exactly as in Art. 8, it is seen that the symbolical form of an x operator equal to this is $\eta\, d/d\eta \log(\xi/\eta)$, and that its expanded form is obtained by omitting the first term, and then putting $n = 0$ in the general value (33). Thus, the operator which has to be defined as $\{0, 1;\ 0, 0\}_x$ is

$$\frac{x_2}{x_1}\frac{d}{dx_1} + \frac{\mathrm{G}x_2}{x_1^2}\frac{d}{dx_2} + \frac{\mathrm{G}^2x_2}{2!\,x_1^3}\frac{d}{dx_3} + \ldots, \qquad (50)$$

where G is the generator defined in (32A).

13. Examples of important operators which occur among those transformed in the last two articles are before us in (48) and in the equality of (49) and (50). It is unnecessary to multiply particular instances as they can be deduced without number by giving m particular integral values in (43) . . . (46). It is to be noticed that, excluding the special cases of $m = 0$ and $m = 1$, one or other of the two equivalent operators will involve as coefficients those in a multinomial expansion of negative index. Thus, for instance,

$$2\{1, 0;\ 2, -2\}_x = -\{0, 1;\ -1, 1\}_y + y_1^{-2}\{0, 1;\ 1, -1\}_y,$$

a particular case of (43), is at more length

$$\begin{aligned} &2x_1x_2\frac{d}{dx_1} + (2x_1x_3 + x_2^2)\frac{d}{dx_2} + (2x_1x_4 + 2x_2x_3)\frac{d}{dx_3} + \ldots \\ &= -\left\{\frac{y_2^2 - y_1y_3}{y_1^3}\frac{d}{dy_2} - 2\,\frac{(y_3^3 - 2y_1y_2y_3 + y_1^2y_4)}{y_1^4}\frac{d}{dy_3} + \ldots\right\} \\ &\quad + \frac{1}{y_1^2}\left\{2y_2\frac{d}{dy_1} + 3y_3\frac{d}{dy_2} + 4y_4\frac{d}{dy_3} + \ldots\right\}. \qquad (51) \end{aligned}$$

The transformation of the operator G of (32A) is an application of (49). Thus

$$\begin{aligned} \mathrm{G}(x, y) &= x_1\left(x_2\frac{d}{dx_1} + 2x_3\frac{d}{dx_2} + 3x_4\frac{d}{dx_3} + \ldots\right) - x_2\left(x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2} + x_3\frac{d}{dx_3} + \ldots\right) \\ &= x_1\{-1, 1;\ 1, -1\}_x - x_2\{1, 0;\ 1, 0\}_x \\ &= y_1^{-1}\{0, 1;\ 0, 0\}_y - y_1^{-3}y_2\{0, 1;\ 1, 0\}_y, \end{aligned}$$

by (49) and (23),

$$\begin{aligned} &= y_1^{-1}\left\{\frac{y_2}{y_1}\frac{d}{dy_1} + \frac{\mathrm{G}y_2}{y_1^2}\frac{d}{dy_2} + \frac{\mathrm{G}^2y_2}{2!\,y_1^3}\frac{d}{dy_3} + \ldots\right\} \\ &\quad - y_1^{-3}y_2\left\{y_1\frac{d}{dy_1} + 2y_2\frac{d}{dy_2} + 3y_3\frac{d}{dy_3} + \ldots\right\} \\ &= \frac{2y_1y_3 - 3y_2^2}{y_1^3}\frac{d}{dy_2} + \frac{3y_1^2y_4 - 6y_1y_2y_3 + y_2^3}{y_1^4}\frac{d}{dy_3} + \ldots \qquad (52) \end{aligned}$$

14. Operators for which $m + n$ is negative still remain to be considered. In particular, those of the type $\{0, 1; 0, -n'\}$ have still to be defined. About the right definition of them there can, however, after articles 8 and 12, be no doubt.

The general principle by means of which if $\{\mu, \nu; m, n\}_y$ is known, the form of $\{\mu, \nu; m, n-r\}_y$ is deduced is expressed by the rule—"Write $\{\mu, \nu, m, n\}_y$ symbolically, by putting ξ^p for each d/dy_p, divide through by ξ^r, reject all terms, if any now occur, which do not contain a positive power of ξ as factor, and then for each ξ^p write d/dy_p."

Thus to accord with (33) and (50) the right operator to be defined as

$$\{0, 1; 0, -n'\}_y,$$

where $-n'$ is a negative integer, is

$$\frac{G^{n'}y_2}{n'!\,y_1^{n'+1}}\cdot\frac{d}{dy_1}+\frac{G^{n'+1}y_2}{(n'+1)!\,y_1^{n'+2}}\cdot\frac{d}{dy_2}+\frac{G^{n'+2}y_2}{(n'+2)!\,y_1^{n'+3}}\cdot\frac{d}{dy_3}+\ldots \qquad (53)$$

We now proceed to the transformation of $\{\mu, \nu; m, -m-r\}_x$ where r is a positive integer.

The symbolical form of $m\{1, 0; m, -m-r\}_x$ is as in Arts. 6 and 11

$$\eta^{-m-r}\{\xi^m - X_m^{(m)}\eta^m - X_{m+1}^{(m)}\eta^{m+1} - \ldots - X_{m+r}^{(m)}\eta^{m+r}\}. \quad . \quad . \quad . \quad . \quad (54)$$

The symbolical form of its transformation is, therefore,

$$-\xi^m\eta^{-m-r}\frac{d\eta}{d\xi}+X_m^{(m)}\eta^{-r}\frac{d\eta}{d\xi}+X_{m+1}^{(m)}\eta^{-r+1}\frac{d\eta}{d\xi}+\ldots+X_{m+r}^{(m)}\frac{d\eta}{d\xi}. \quad . \quad (55)$$

This when expanded in terms of ξ can involve no zero or negative powers. For it is a sum of multiples of $\eta\, d\eta/d\xi$, $\eta^2\, d\eta/d\xi$, . . . only, since $\{1, 0; m, -m-r\}_x$ is a sum of multiples of η, η^2, . . . only, and these when expressed in terms of ξ are all free from ξ^0, ξ^{-1} ξ^{-2} . . . Thus the coefficients of ξ^{-r}, ξ^{-r+1}, . . . 1, which would appear to occur in the above symbolical form of the transformation of $\{1, 0; m, -m-r\}$ are in reality absent; and, consequently,

$$m\{1, 0; m, -m-r\}_x = -\{0, 1; 1-m-r, m-1\}_y + X_m^{(m)}\{0, 1; 1-r, -1\}_y$$
$$+X_{m+1}^{(m)}\{0, 1; 2-r, -1\}_y + \ldots + X_{m+r-1}^{(m)}\{0, 1; 0, -1\}_y + X_{m+r}^{(m)}\{0, 1; 1, -1\}_y, \quad (56)$$

the various terms on the right consisting of the parts with positive indices of ξ from the corresponding terms of (55).

In like manner $m\{0, 1; m, -m-r\}_x$ whose symbolical form is

$$\eta^{1-m-r}\frac{d}{d\eta}\{\xi^m - \mathrm{X}_m^{(m)}\eta^m - \mathrm{X}_{m+1}^{(m)}\eta^{m+1} - \ldots - \mathrm{X}_{m+r}^{(m)}\eta^{m+r}\}, \quad . \quad . \quad . \quad (57)$$

a form proceeding by positive integral powers of η, and therefore of ξ, transforms into

$$-m\xi^{m-1}\eta^{1-m-r} + m\mathrm{X}_m^{(m)}\eta^{-r}\frac{d\eta}{d\xi} + (m+1)\,\mathrm{X}_{m+1}^{(m)}\eta^{1-r}\frac{d\eta}{d\xi} + \ldots$$
$$+ (m+r-1)\,\mathrm{X}_{m+r-1}^{(m)}\eta^{-1}\frac{d\eta}{d\xi} + (m+r)\,\mathrm{X}_{m+r}^{(m)}\frac{d\eta}{d\xi}, \quad . \quad (58)$$

of which the terms in zero and negative powers of ξ must, as before, disappear, leaving as the result of transformation

$$m\{0, 1; m, -m-r\}_x = -m(1-m-r)\{1, 0; 1-m-r, m-1\}_y$$
$$+ m\mathrm{X}_m^{(m)}\{0, 1; 1-r, -1\}_y + (m+1)\mathrm{X}_{m+1}^{(m)}\{0, 1; 2-r, -1\}_y + \ldots$$
$$+ (m+r-1)\mathrm{X}_{m+r-1}^{(m)}\{0, 1; 0, -1\}_y + (m+r)\mathrm{X}_{m+r}^{(m)}\{0, 1; 1, -1\}_y. \quad (59)$$

By addition of μ times (56) to ν times (59) the transformation of the more general $\{\mu, \nu; m, -m-r\}_x$ is at once deduced.

15. The forms taken by (56) and (59) for the case $r = 1$, *i.e.*, $m + n = -1$, since

$$\mathrm{X}_m^{(m)} = x_1^m = y_1^{-m}$$

and

$$\mathrm{X}_{m+1}^{(m)} = mx_1^{m-1}x_2 = -my_1^{-m-2}y_2,$$

may be written

$$m\{1, 0; m, -m-1\}_x = -\{0, 1; -m, m-1\}_y + y_1^{-m}\{0, 1; 0, -1\}_y$$
$$- my_1^{-m-2}y_2\{0, 1; 1, -1\}_y. \quad (60)$$

and

$$\{0, 1; m, -m-1\}_x = m\{1, 0; -m, m-1\}_y + y_1^{-m}\{0, 1; 0, -1\}_y$$
$$-(m+1)\,y_1^{-m-2}y_2\{0, 1; 1, -1\}_y. \quad (61)$$

One or two particular cases of these formulæ deserve mention. The value zero of m makes (60) an identity. In (61) the substitution of the same value produces

$$\{0, 1; 0, -1\}_x = \{0, 1; 0, -1\}_y - y_1^{-2}y_2\{0, 1; 1, -1\}_y,$$

in verification of which we may notice that it is unaltered by interchange of x and y, in virtue of (48). Another way of writing the result is to say that

$$2\{0, 1;\ 0, -1\}_x - x_1^{-2} x_2 \{0, 1;\ 1, -1\}_x \\ = 2\{0, 1;\ 0, -1\}_y - y_1^{-2} y_2 \{0, 1;\ 1, -1\}_y \quad . \quad (62)$$

is a self reciprocal operator of positive character.

We are now enabled to write (61)

$$m\{0, 1; m, -m-1\}_x \\ = m\{1, 0;\ -m, m-1\}_y + y_1^{-m}\{0, 1;\ 0, -1\}_x - m y_1^{-m-2} y_2 \{0, 1;\ 1, -1\}_y,$$

which becomes (60) upon interchanging x and y, replacing m by $-m$, and using (48) and the values for x_1 and x_2, in terms of y_1 and y_2. Thus we have another verification of the consistency of our results.

II. *Ternary Operators.*

16. Let x, y, z be three variables connected by a relation of any form known or unknown. Let x_{rs}, y_{rs}, z_{rs} denote respectively

$$\frac{1}{r\,!\,s\,!}\frac{d^{r+s}x}{dy^r\,dz^s}, \qquad \frac{1}{r\,!\,s\,!}\frac{d^{r+s}y}{dz^r\,dx^s}, \qquad \frac{1}{r\,!\,s\,!}\frac{d^{r+s}z}{dx^r\,dy^s}.$$

Let ξ, η, ζ be any set of corresponding increments of x, y, z. They are connected by a single relation, which may be written in either of the forms

$$\xi = (x_{10}\,\eta + x_{01}\,\zeta) + (x_{20}\,\eta^2 + x_{11}\,\eta\zeta + x_{02}\,\zeta^2) \\ + (x_{30}\,\eta^3 + x_{21}\,\eta^2\,\zeta + x_{12}\,\eta\zeta^2 + x_{03}\,\zeta^3) + \ldots, \quad . \quad . \quad (63)$$

$$\eta = (y_{10}\,\zeta + y_{01}\,\xi) + (y_{20}\,\zeta^2 + y_{11}\,\zeta\xi + y_{02}\,\xi^2) \\ + (y_{30}\,\zeta^3 + y_{21}\,\zeta^2\xi + y_{12}\,\zeta\xi^2 + y_{03}\,\xi^3) + \ldots, \quad . \quad . \quad (64)$$

$$\zeta = (z_{10}\,\xi + z_{01}\,\eta) + (z_{20}\,\xi^2 + z_{11}\,\xi\eta + z_{02}\,\eta^2) \\ + (z_{30}\,\xi^3 + z_{21}\,\xi^2\eta + z_{12}\,\xi\eta^2 + z_{03}\,\eta^3) + \ldots. \quad . \quad . \quad (65)$$

Let m be a positive integer, and let $X_{rs}^{(m)}$ denote the coefficient of $\eta^r\,\zeta^s$ in ξ^m when expanded in ascending products of positive integral powers of η and ζ, so that

$$\xi^m = \{\Sigma x_{pq} \eta^p \zeta^q\}^m = \sum_{r+s \nless m} X_{rs}^{(m)} \eta^r \zeta^s; \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

and in like manner write

$$\eta^m = \{\Sigma y_{pq} \zeta^p \xi^q\}^m = \sum_{r+s \nless m} Y_{rs}^{(m)} \zeta^r \xi^s, \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

and

$$\zeta^m = \{\Sigma z_{pq} \xi^p \eta^q\}^m = \sum_{r+s \nless m} Z_{rs}^{(m)} \xi^r \eta^s. \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

We may include, if we please, the value zero of m; but the expansions of ξ^o, η^o, ζ^o consist only of the single terms $\eta^o \zeta^o$, $\zeta^o \xi^o$, $\xi^o \eta^o$.

The operators to be considered and transformed are the following :—

$$m\{\mu, \nu, \nu'; m, n, n'\}_x = \Sigma (\mu + \nu r + \nu' s) X_{rs}^{(m)} \frac{d}{dx_{n+r, n'+s}}, \quad . \quad . \quad . \quad (69)$$

$$m\{\mu, \nu, \nu'; m, n, n'\}_y = \Sigma (\mu + \nu r + \nu' s) Y_{rs}^{(m)} \frac{d}{dy_{n+r, n'+s}}, \quad . \quad . \quad . \quad (70)$$

$$m\{\mu, \nu, \nu'; m, n, n'\}_z = \Sigma (\mu + \nu r + \nu' s) Z_{rs}^{(m)} \frac{d}{dz_{n+r, n'+s}}; \quad . \quad . \quad . \quad (71)$$

where μ, ν, ν' are any numerical quantities,

m a positive integer,

n, n' positive integers or one or both zero, and

r, s quantities which take in succession all zero and positive integral values subject to $r + s \nless m$.

Cases of m negative, and of n, n' either or both less than -1, which have been dealt with in the analogous theory of binary operators, will not be here considered.

The cases of m zero, and of n or n' equal to -1, will not be entirely excluded, but will be only dealt with as far as their accordance with the results for m positive and n, n' not negative needs no elaboration to make it clear.

Thus our field of investigation *is narrower* than in that of the analogous theory hitherto considered. Were negative values of n and n' admitted, the lower limit of r in the operators (69) . . . (71) would be $-n+1$ instead of zero, and that of s would be in like manner $-n'+1$. Thus when we admit the value -1 of n we must exclude the value 0 of r, and when the value -1 of n' we must exclude the value 0 of s.

Let us now express (69), (70), (71) symbolically as follows :—

$$m\{\mu, \nu, \nu'; m, n, n'\}_x = \Sigma (\mu + \nu r + \nu' s) X_{rs}^{(m)} \eta^{n+r} \zeta^{n'+s}, \quad . \quad . \quad . \quad (72)$$

$$m\{\mu, \nu, \nu'; m, n, n'\}_y = \Sigma (\mu + \nu r + \nu' s) Y_{rs}^{(m)} \zeta^{n+r} \xi^{n'+s}, \quad . \quad . \quad . \quad (73)$$

$$m\{\mu, \nu, \nu'; m, n, n'\}_z = \Sigma (\mu + \nu r + \nu' s) Z_{rs}^{(m)} \xi^{n+r} \eta^{n'+s}; \quad . \quad . \quad . \quad (74)$$

i.e., let us in any x-operator symbolize d/dx_{pq} by $\eta^p \zeta^q$, in any y-operator d/dy_{pq} by $\zeta^p \xi^q$, and in any z-operator d/dz_{pq} by $\xi^p \eta^q$.

We may in this way write (71) or (74)

$$m\{\mu, \nu, \nu'; m, n, n'\}_z = \mu\xi^n \eta^{n'} \{z_{10}\xi + z_{01}\eta + z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2 + \ldots\}^m$$

$$+ \nu\zeta^{n+1}\eta^{n'} \frac{d}{d\xi}\{z_{10}\xi + z_{01}\eta + z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2 + \ldots\}^m$$

$$+ \nu'\xi^n \eta^{n'+1} \frac{d}{d\eta}\{z_{10}\xi + z_{01}\eta + z_{20}\xi^2 + z_{11}\xi\eta + z_{02}\eta^2 + \ldots\}^m$$

$$= \mu\xi^n\eta^{n'}\zeta^m + \nu\xi^{n+1}\eta^{n'}\frac{d}{d\xi}(\zeta^m) + \nu'\xi^n\eta^{n'+1}\frac{d}{d\eta}(\zeta^m), \quad . \quad . \quad (75)$$

where ζ means the expansion in terms of ξ and η given in (65), and where the symbolization denotes that the right-hand member is to be expanded in terms of ξ and η, and to have each product $\xi^p \eta^q$ in its expansion replaced by the corresponding d/dz_{pq}, in order to produce the operator in z dependent which is represented by the notation on the left.

Thus in particular, assigning to different pairs in succession of the three parameters, μ, ν, ν', zero values,

$$m\{1, 0, 0; m, n, n'\}_z = \xi^n \eta^{n'} \zeta^m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

$$m\{0, 1, 0; m, n, n'\}_z = \xi^{n+1}\eta^{n'}\frac{d}{d\xi}(\zeta^m) = m\xi^{n+1}\eta^{n'}\zeta^{m-1}\frac{d\zeta}{d\xi}, \quad . \quad . \quad . \quad (77)$$

$$m\{0, 0, 1; m, n, n'\}_z = \xi^n\eta^{n'+1}\frac{d}{d\eta}(\zeta^m) = m\xi^n\eta^{n'+1}\zeta^{m-1}\frac{d\zeta}{d\eta}, \quad . \quad . \quad . \quad (78)$$

while

$$\{\mu, \nu, \nu'; m, n, n'\}_z = \mu\{1, 0, 0; m, n, n'\}_z + \nu\{0, 1, 0; m, n, n'\}_z$$
$$+ \nu'\{0, 0, 1; m, n, n'\}_z. \quad . \quad . \quad (79)$$

Precisely similar symbolical expressions to (75) . . . (78) are, of course, assigned to the corresponding operators in x and in y dependent. We have only cyclically to interchange ξ, η, ζ once and twice respectively, and to regard the expressions on the right thus obtained as short ways of writing their expansions by aid of (63) and (64) in terms of η, ζ and ζ, ξ respectively.

17. In the present article the expression of each operative symbol d/dx_{rs}, on a function of the derivatives of x with regard to y and z, in terms of the operative symbols d/dz_{pq} on the equivalent function of the derivatives of z with regard to x and y, is investigated.

If, as in the earlier part of the last article, ξ, η, ζ are simultaneous increments of x, y, z, we may look upon

$$x_{10},\ x_{01},\ x_{20},\ x_{11},\ x_{02},\ \ldots$$

as a number of independent quantities; upon

$$y_{10},\ y_{01},\ y_{20},\ y_{11},\ y_{02},\ \ldots$$

and

$$z_{10},\ z_{01},\ z_{20},\ z_{11},\ z_{02},\ \ldots$$

as determinate functions of these quantities; upon ξ, η, ζ as three quantities connected with one another, and with x_{10}, x_{01}, x_{20}, ... by a relation of which (63), (64), and (65) are equivalent forms.

Of the quantities x_{10}, x_{01}, x_{20}, ... let one x_{rs} alone receive an infinitesimal variation: also of ξ, η, ζ, let η and ζ be kept constant so that ξ receives a consequent variation. Some or all of y_{10}, y_{01}, y_{20}, ... and some or all of z_{10}, z_{01}, z_{20}, ... will also receive consequent variations. From (63) we thus obtain

$$\delta\xi = \eta^r\zeta^s\,\delta x_{rs};$$

from (64)

$$0 = \{y_{01} + y_{11}\xi + 2y_{02}\eta + y_{21}\xi^2 + 2y_{12}\xi\eta + 3y_{03}\eta^2 + \ldots\}\,\delta\xi$$
$$+ \left\{\frac{dy_{10}}{dx_{rs}}\zeta + \frac{dy_{01}}{dx_{rs}}\xi + \frac{dy_{20}}{dx_{rs}}\zeta^2 + \frac{dy_{11}}{dx_{rs}}\zeta\xi + \frac{dy_{02}}{dx_{rs}}\xi^2 + \ldots\right\}\delta x_{rs};$$

and from (65)

$$0 = \{z_{10} + 2z_{20}\xi + z_{11}\eta + 3z_{30}\xi^2 + 2z_{21}\xi\eta + z_{12}\eta^2 \ldots\}\,\delta\xi$$
$$+ \left\{\frac{dz_{10}}{dx_{rs}}\xi + \frac{dz_{01}}{dx_{rs}}\eta + \frac{dz_{20}}{dx_{rs}}\xi^2 + \frac{dz_{11}}{dx_{rs}}\xi\eta + \frac{dz_{02}}{dx_{rs}}\eta^2 + \ldots\right\}\delta x_{rs}.$$

The three relations are identical. Let us study the identity of the first and third We obtain from them that

$$\frac{dz_{10}}{dx_{rs}}\xi + \frac{dz_{01}}{dx_{rs}}\eta + \frac{dz_{20}}{dx_{rs}}\xi^2 + \frac{dz_{11}}{dx_{rs}}\xi\eta + \frac{dz_{02}}{dx_{rs}}\eta^2 + \ldots$$
$$= -\eta^r\zeta^s\{z_{10} + 2z_{20}\xi + z_{11}\eta + 3z_{30}\xi^2 + 2z_{21}\xi\eta + z_{12}\eta^2 + \ldots\}, \quad . \quad . \quad (80)$$

for all values of ξ and η; and, consequently, that if by aid of (65) the right hand member be like the left, expanded in powers and products of powers of ξ and η, the coefficients of corresponding terms on the two sides will be equal. In other words, each dz_{pq}/dx_{rs} is the coefficient of the corresponding $\xi^p\eta^q$.

Now, in the equivalence of operators,

$$\frac{d}{dx_{rs}} = \frac{dz_{10}}{dx_{rs}} \cdot \frac{d}{dz_{10}} + \frac{dz_{01}}{dx_{rs}} \cdot \frac{d}{dx_{01}} + \frac{dz_{20}}{dx_{rs}} \cdot \frac{d}{dz_{20}} + \frac{dz_{11}}{dx_{rs}} \cdot \frac{d}{dz_{11}} + \frac{dz_{02}}{dx_{rs}} \cdot \frac{d}{dz_{02}} + \dots$$

each dz_{pq}/dx_{rs} is the coefficient of d/dz_{pq}.

It follows that in the expansion in terms of ξ and η of the right hand member of (80) the substitution for each $\xi^p\eta^q$ of the corresponding d/dz_{pq} exactly produces the expression for d/dx_{rs}. In other words, for each r and s, the

$$z \text{ transform of } \frac{d}{dx_{rs}} = -\eta^r\zeta^s\frac{d\zeta}{d\xi}, \quad \dots \dots \quad (81)$$

where ζ and its partial differential coefficient are to be replaced by their equivalents in terms of ξ and η by (65), where the product is to be expanded in terms of ξ and η, and where in the expanded result each product $\xi^p\eta^q$ is to be replaced by the corresponding operative symbol d/dz_{pq}.

By (77) we see that d/dx_{rs} is thus replaced by a linear z-operator of the form under consideration; in fact that

$$\frac{d}{dx_{rs}} = -\{0, 1, 0; s+1, -1, r\}_z. \quad \dots \dots \quad (82)$$

Since the μ and the ν' of this operator are zero, the fact that n is -1 gives no difficulty as to the presence or absence of coefficients on the right like $Z_{0q}^{(s+1)}$.

In precisely the same way, by giving variations to ξ and x_{rs} in (63) and (64) instead of (63) and (65), we might have obtained

$$\frac{d}{dx_{rs}} = -\{0, 0, 1; r+1, s, -1\}_y, \quad \dots \dots \quad (83)$$

of which y-operator the symbolical form is

$$-\zeta^s\eta^r\frac{d\eta}{d\xi}.$$

18. The rules for transforming any linear x-operator to its equivalent forms in y and in z dependent, are now very simply expressed just as was the analogous rule in in Art. 4. Since the x-operator

$$\frac{d}{dx_{rs}} \quad \text{or} \quad \eta^r\zeta^s$$

has for its equivalent y-operator

$$-\eta^r\zeta^s\frac{d\eta}{d\xi},$$

and for its equivalent z-operator

$$-\eta^r\zeta^s\frac{d\zeta}{d\xi},$$

these rules are merely—To find the equivalent y-operator to a given linear x-operator, multiply its symbolical form by $-d\eta/d\xi$; and to find its equivalent z-operator, multiply that symbolical form by $-d\zeta/d\xi$. The y-operator thus obtained has of course to be expanded in terms of ζ and ξ by (64), and the z-operator in terms of ξ and η by (65), before being intelligible except by means of (76) to (78).

In verification let it be noticed that, since by first principles of the theory of partial differentiation the three sets of ratios

$$\frac{d\xi}{d\eta} : \frac{d\xi}{d\zeta} : -1,$$

$$-1 : \frac{d\eta}{d\zeta} : \frac{d\eta}{d\xi},$$

$$\frac{d\zeta}{d\eta} : -1 : \frac{d\zeta}{d\xi},$$

are equal, precisely the same results are obtained by cyclical interchange of x, y, z and ξ, η, ζ.

19. Now as in (76)

$$m\{1, 0, 0; m, n, n'\}_x = \eta^n\zeta^{n'}\xi^m.$$

Its forms in y and z respectively are then

$$-\zeta^{n'}\xi^m\eta^n\frac{d\eta}{d\xi}, \quad \text{and} \quad -\xi^m\eta^n\zeta^{n'}\frac{d\zeta}{d\xi}.$$

Of these by two cyclic interchanges in (78), and by (77) itself, respectively, the expressions are

$$-\{0, 0, 1; n+1, n', m-1\}_y, \quad \text{and} \quad -\{0, 1, 0; n'+1, m-1, n\}_z$$

consequently

$$m\{1, 0, 0; m, n, n'\}_x = -\{0, 0, 1; n+1, n', m-1\}_y$$
$$= -\{0, 1, 0; n'+1, m-1, n\}_z. \quad . \quad (84)$$

In the same way the y and z transforms of

$$\{0, 1, 0; m, n, n'\}_x, \quad i.e., \quad \eta^{n+1}\zeta^{n'}\xi^{m-1}\frac{d\xi}{d\eta},$$

are

$$-\zeta^{n'}\xi^{m-1}\eta^{n+1}\frac{d\xi}{d\eta}\cdot\frac{d\eta}{d\xi}, \quad \text{and} \quad -\xi^{m-1}\eta^{n+1}\zeta^{n'}\frac{d\xi}{d\eta}\cdot\frac{d\zeta}{d\xi},$$

i.e.,

$$-\zeta^{n'}\xi^{m-1}\eta^{n+1}, \quad \text{and} \quad +\xi^{m-1}\eta^{n+1}\zeta^{n'}\frac{d\zeta}{d\eta},$$

by the equalities of ratios at the end of the last article. Accordingly

$$\{0, 1, 0; m, n, n'\}_x = -(n+1)\{1, 0, 0; n+1, n', m-1\}_y$$
$$= \{0, 0, 1; n'+1, m-1, n\}_z. \quad . \quad (85)$$

And once more, precisely in the same way,

$$\eta^n \zeta^{n'+1} \xi^{m-1} \frac{d\xi}{d\zeta}, \quad \zeta^{n'+1} \xi^{m-1} \eta^n \frac{d\eta}{d\zeta}, \quad \text{and} \quad -\xi^{m-1} \eta^n \zeta^{n'+1}$$

are equivalent operators in x, y, z respectively dependent, so that also

$$\{0, 0, 1; m, n, n'\}_x = \{0, 1, 0; n+1, n', m-1\}_y$$
$$= -(n'+1)\{1, 0, 0; n'+1, m-1, n\}_z. \quad . \quad (86)$$

Of these sets of equalities (85) and (86) may in reality be deduced from (84) by cyclical interchanges of the variables and alteration of parameters. The independent investigation above is justified by the verification it affords.

The general formulæ of transformation, including (84), (85), (86), follow from them by (79), and are

$$\{\mu, \nu, \nu'; m, n, n'\}_x = \{-\nu(n+1), \nu', -\frac{\mu}{m}; n+1, n', m-1\}_y$$

$$= \{-\nu'(n'+1), -\frac{\mu}{m}, \nu; n'+1, m-1, n\}_z. \quad . \quad (87)$$

Included, it is interesting to notice that we have three distinct classes of self reproductive or cyclically persistent operators, of characters corresponding each to one of the cube roots of unity, viz.:

$$\{-m, 1, 1; m, m-1, m-1\}_x = \{-m, 1, 1; m, m-1, m-1\}_y$$
$$= \{-m, 1, 1; m, m-1, m-1\}_z \quad . \quad . \quad (88)$$

$$\{-m, \omega, \omega^2; m, m-1, m-1\}_x = \omega\{-m, \omega, \omega^2; m, m-1, m-1\}_y$$
$$= \omega^2\{-m, \omega, \omega^2; m, m-1, m-1\}_z. \quad (89)$$

$$\{-m, \omega^2, \omega; m, m-1, m-1\}_x = \omega^2\{-m, \omega^2, \omega; m, m-1, m-1\}_y$$
$$= \omega\{-m, \omega^2, \omega; m, m-1, m-1\}_z. \quad (90)$$

20. Some of the simplest, and most important so far as actual experience goes, examples of the formulæ now proved will be considered in what follows.

The only lineo-linear operators, of the classes with which we are dealing, both of whose cyclical transformations are also lineo-linear, are found by putting m, $n+1$ and $n'+1$ all equal to unity in the results of the last article. Thus

$$\{1, 0, 0; 1, 0, 0\}_x = -\{0, 0, 1; 1, 0, 0\}_y = -\{0, 1, 0; 1, 0, 0\}_z, \quad . \quad (91)$$

with the correlative equalities obtained by writing y, z, x and z, x, y respectively for x, y, z, involve the aggregate of all such operators. At length the equalities (91) are

$$x_{10}\frac{d}{dx_{10}} + x_{01}\frac{d}{dx_{01}} + x_{20}\frac{d}{dx_{20}} + x_{11}\frac{d}{dx_{11}} + x_{02}\frac{d}{dx_{02}} + x_{30}\frac{d}{dx_{30}} + x_{21}\frac{d}{dx_{21}} + x_{12}\frac{d}{dx_{12}} + x_{03}\frac{d}{dx_{03}} + \ldots$$

$$= -\left\{y_{01}\frac{d}{dy_{01}} + y_{11}\frac{d}{dy_{11}} + 2y_{02}\frac{d}{dy_{02}} + y_{21}\frac{d}{dy_{21}} + 2y_{12}\frac{d}{dy_{12}} + 3y_{03}\frac{d}{dy_{03}} + \ldots\right\}$$

$$= -\left\{z_{10}\frac{d}{dz_{10}} + 2z_{20}\frac{d}{dz_{20}} + z_{11}\frac{d}{dz_{11}} + 3z_{30}\frac{d}{dz_{30}} + 2z_{21}\frac{d}{dz_{21}} + z_{12}\frac{d}{dz_{12}} + \ldots\right\} \quad . \quad (92)$$

We thus learn that, if a function of the derivatives of x with regard to y and z is homogeneous, the equivalent function of the derivatives of y with regard to z and x is isobaric in second suffixes, while the equivalent function of the derivatives of z with regard to x and y is isobaric in first suffixes; and that

$$i(x, yz) = -w_2(y, zx) = -w_1(z, xy), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

where the notation explains itself. The correlative facts are

$$-w_1(x, yz) = i(y, zx) = -w_2(z, xy), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

and

$$-w_2(x, yz) = -w_1(y, zx) = i(z, xy). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

The same aggregate as is involved in (91) and its correlatives is also expressed by the facts that

$$\{-1, 1, 1; 1, 0, 0\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

$$\{-1, \omega, \omega^2; 1, 0, 0\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

$$\{-1, \omega^2, \omega; 1, 0, 0\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

obtained by giving m the value 1 in (88) to (90), are cyclically persistent lineo-linear operators of characters $1, \omega, \omega^2$ respectively.

If the operation be on a homogeneous and doubly isobaric function we are thus told that

$$-i + w_1 + w_2 \qquad -i + \omega w_1 + \omega^2 w_2, \qquad -i + \omega^2 w_1 + \omega w_2 \quad . \quad . \quad (99)$$

are characteristics which persist after one cyclical transformation but for the multipliers 1, ω, ω^2 respectively, and after a second but for 1, ω^2, ω.

21. The quadro-linear operators (linear operators with coefficients quadratic in the derivatives) both whose cyclic transformations are also quadro-linear, are obtained by giving to every one of m, $n + 1$, $n' + 1$ the value 2 in the formulæ of Art. 19. Their aggregate is involved in

$$2\{1, 0, 0;\ 2, 1, 1\}_x = -\{0, 0, 1;\ 2, 1, 1\}_y = -\{0, 1, 0;\ 2, 1, 1\}_z, \quad . \quad (100)$$

and its two correlatives in y, z, x and z, x, y.

The same system is expressed by the three cyclically persistent quadro-linear operators

$$\{-2, 1, 1;\ 2, 1, 1\}, \text{ of character } 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

$$\{-2, \omega, \omega^2;\ 2, 1, 1\}, \qquad ,, \qquad \omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

$$\{-2, \omega^2, \omega;\ 2, 1, 1\}, \qquad ,, \qquad \omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

Of these the first expanded to a few terms is

$$X_{30}^{(2)} \frac{d}{dx_{41}} + X_{21}^{(2)} \frac{d}{dx_{32}} + X_{12}^{(2)} \frac{d}{dx_{23}} + X_{03}^{(2)} \frac{d}{dx_{14}} + 2\left(X_{40}^{(2)} \frac{d}{dx_{51}} + \ldots\right)$$
$$+ 3\left(X_{50}^{(2)} \frac{d}{dx_{61}} + \ldots\right) + \ldots, \quad . \quad . \quad (104)$$

where

$$X_{20}^{(2)} = x_{10}{}^2,\ X_{11}^{(2)} = 2x_{10}x_{01},\ X_{02}^{(2)} = x_{01}{}^2,$$

$$X_{30}^{(2)} = 2x_{10}x_{20},\ X_{21}^{(2)} = 2x_{10}x_{11} + 2x_{01}x_{20},\ X_{12}^{(2)} = 2x_{10}x_{02} + 2x_{01}x_{11},\ X_{03}^{(2)} = 2x_{01}x_{02},$$

$$X_{40}^{(2)} = 2x_{10}x_{30} + x_{20}{}^2,\ X_{31}^{(2)} = 2x_{10}x_{21} + 2x_{01}x_{12} + 2x_{20}x_{11},\ X_{22}^{(2)} = 2x_{10}x_{12} + 2x_{01}x_{21} + x_{11}{}^2$$
$$+ 2x_{20}x_{02},\ X_{13}^{(2)} = 2x_{10}x_{03} + 2x_{01}x_{12} + 2x_{11}x_{02},\ X_{04}^{(2)} = 2x_{01}x_{03} + x_{02}{}^2,$$

$$X_{50}^{(2)} = 2x_{10}x_{40} + 2x_{20}x_{30}, \ldots,$$

and generally

$$X_{mn}^{(2)} = \sum_{r+s \nless 1}^{r+s \ngtr m+n-1} (x_{rs}x_{m-r,\,n-s}).$$

The two imaginary cyclically persistent quadro-linear operators (102) and (103) are easily written out in like manner. They commence with terms in d/dx_{31}, d/dx_{22}, d/dx_{13}, which it is to be observed are wanting from the above.

Once more by giving to each of m, $n+1$, $n'+1$ the value 3 in Art. 19, an aggregate is obtained of linear operators with coefficients of the third degree, whose transforms have both of them coefficients of the third degree also. The aggregate may, as before, be considered involved in three cyclically persistent operators of the type, one of each character. Similarly as to operators with coefficients of any higher degree.

22. Some of the most important linear operators which have been used in recent theories of functional invariants, cyclicants, &c., have the property of persistence of degree in the derivatives after one cyclical transformation, but not after a second.

Such operators occur among those obtained by putting $n+1=m$ in (87), viz.,

$$\{\mu, \nu, \nu'; m, m-1, n'\}_x = \left\{-\nu m, \nu', -\frac{\mu}{m}; m, n', m-1\right\}_y$$
$$= \left\{-\nu'(n'+1), -\frac{\mu}{m}, \nu; n'+1, m-1, m-1\right\}_z \quad . \quad . \quad . \quad (105)$$

In particular, there are three classes of operators which have a property closely akin to that of persisting in form after a first cyclical transformation, being, in fact, only altered by the interchange of first and second suffixes: they are

$$\{-m, 1, 1; m, m-1, n'\}_x = \{-m, 1, 1; m, n', m-1\}_y$$
$$= \{-(n'+1), 1, 1; n'+1, m-1, m-1\}_z \quad . \quad . \quad . \quad . \quad (106)$$

$$\{-m, \omega, \omega^2; m, m-1, n'\}_x = \omega\{-m, \omega, \omega^2; m, n', m-1\}_y$$
$$= \omega^2\{-(n'+1), \omega, \omega^2; n'+1, m-1, m-1\}_z \quad . \quad . \quad . \quad (107)$$

$$\{-m, \omega^2, \omega; m, m-1, n'\}_x = \omega^2\{-m, \omega^2, \omega; m, n', m-1\}_y$$
$$= \omega\{-(n'+1), \omega^2, \omega; n'+1, m-1, m-1\}_z \quad . \quad . \quad . \quad (108)$$

It is to be noticed, in the case of the first of these, that the second cyclical transformation, which is of different degree from the first, is quite symmetrical in first and second suffixes.

Among the operators comprised in (106) occur the two, which I have called ω_1 and ω_2,* two of the six form annihilators of projective cyclicants, viz.,

$$\omega_1(x, yz) = \sum_{r+s \nless 1} \left\{(r+s-1)\, x_{rs} \frac{d}{dx_{r,\, s+1}}\right\},$$

$$\omega_2(x, yz) = \sum_{r+s \nless 1} \left\{(r+s-1)\, x_{rs} \frac{d}{dx_{r+1,\, s}}\right\},$$

* 'London Math. Soc. Proc.,' vol. 20, pp. 131–160.

or in present notation $\{-1, 1, 1; 1, 0, 1\}$ and $\{-1, 1, 1; 1, 1, 0\}$. For the transformation of these we have, by putting 1 for each of m and n' in (106),

$$\omega_1(x, yz) = \omega_2(y, zx) = \{-2, 1, 1; 2, 0, 0\}_z, \quad . \quad . \quad . \quad . \quad (109)$$

of which right hand operator the expansion is

$$Z^{(2)}_{30}\frac{d}{dz_{30}} + Z^{(2)}_{21}\frac{d}{dz_{21}} + Z^{(2)}_{12}\frac{d}{dz_{12}} + Z^{(2)}_{03}\frac{d}{dz_{03}} + 2\left(Z^{(2)}_{40}\frac{d}{dz_{40}} + \ldots\right) + 3\left(Z^{(2)}_{50}\frac{d}{dz_{50}} + \ldots\right) + \ldots$$

where the coefficients have meanings, as in Art. 21.

Closely resembling, but distinct from ω_1 and ω_2, are Mr. FORSYTH'S Δ_2 and Δ_1,* *i.e.*,

$$\Delta_2(x, yz) = \sum_{r+s \not< 1}\left\{(r+s)\,x_{rs}\frac{d}{dx_{r, s+1}}\right\} = \{0, 1, 1; 1, 0, 1\}_x,$$

$$\Delta_1(x, yz) = \sum_{r+s \not< 1}\left\{(r+s)\,x_{rs}\frac{d}{dx_{r+1, s}}\right\} = \{0, 1, 1; 1, 1, 0\}_x.$$

These are also transformed by means of the present article, but have not the property of companionship belonging to ω_1 and ω_2. In fact, by (105),

$$\Delta_2(x, yz) = \{-1, 1, 0; 1, 1, 0\}_y = \{-2, 0, 1; 2, 0, 0\}_z, \quad . \quad . \quad (110)$$

and, by (105), with z, x, y put for x, y, z,

$$\Delta_1(x, yz) = \{-2, 1, 0; 2, 0, 0\}_y = \{-1, 0, 1; 1, 0, 1\}_z. \quad . \quad . \quad (111)$$

23. The special importance of many operators in which the first derivatives do not occur is well known. The form of such operators (in z dependent) is symbolically

$$\xi^p\eta^q\left(a + b\xi\frac{d}{d\xi} + c\eta\frac{d}{d\eta}\right)(\zeta - z_{10}\xi - z_{01}\eta)^m.$$

As every such operator is a sum of multiples of complete operators $\{\mu, \nu, \nu'; m, n, n'\}_z$ so that their theory is implicitly involved in that above discussed, no attempt will be made here to develope it independently. In the present article, however, an interesting class of cyclically persistent operators will be obtained, and a method of procedure in a much wider class of cases will be thus exemplified.

It is required to prove that the result of replacing each first derivative by zero in

$$\{-m, 1, 1; m, 0, 0\}$$

* See his Memoir "A Class of Functional Invariants," 'Phil. Trans.,' A., vol. 180 (1889), pp. 71–118.

is, but for a first derivative factor, an operator which persists in form after one and two cyclical interchanges of the variables.

Symbolically we have, *if square brackets indicate that in an operator first derivatives are thus omitted,*

$$[-m,\,1,\,1\,;\,m,\,0,\,0]_x = \frac{1}{m}\left(-m + \eta\frac{d}{d\eta} + \zeta\frac{d}{d\zeta}\right)(\xi - x_{10}\eta - x_{01}\zeta)^m,$$
$$= (\xi - x_{10}\eta - x_{01}\zeta)^{m-1}\left\{-\xi + x_{10}\eta + x_{01}\zeta + \eta\frac{d\xi}{d\eta} - x_{10}\eta + \zeta\frac{d\xi}{d\zeta} - x_{01}\zeta\right\}$$
$$= (\xi - x_{10}\eta - x_{01}\zeta)^{m-1}\left\{\eta\frac{d\xi}{d\eta} + \zeta\frac{d\xi}{d\zeta} - \xi\right\}.$$

The y transform of this operator is therefore, by Art. 18,

$$-(\xi - x_{10}\eta - x_{01}\zeta)^{m-1}\left\{\eta\frac{d\xi}{d\eta} + \zeta\frac{d\xi}{d\zeta} - \xi\right\}\frac{d\eta}{d\xi},$$

which, since

$$x_{10} : x_{01} : -1 = -1 : y_{10} : y_{01} = z_{01} : -1 : z_{10},$$

and

$$\frac{d\xi}{d\eta} : \frac{d\xi}{d\zeta} : -1 = -1 : \frac{d\eta}{d\zeta} : \frac{d\eta}{d\xi} = \frac{d\zeta}{d\eta} : -1 : \frac{d\zeta}{d\xi},$$

may be written

$$(-1)^{m-1}x_{10}{}^{m-1}(\eta - y_{10}\zeta - y_{01}\xi)^{m-1}\left\{\zeta\frac{d\eta}{d\zeta} + \xi\frac{d\eta}{d\xi} - \eta\right\},$$

and is consequently

$$(-1)^{m-1}x_{10}{}^{m-1}[-m,\,1,\,1\,;\,m,\,0,\,0]_y.$$

In exactly the same way the z transform of the same operator is

$$(-1)^{m-1}x_{01}{}^{m-1}[-m,\,1,\,1\,;\,m,\,0,\,0]_z.$$

Thus we have the formula of transformation

$$[-m,1,1;m,0,0]_x = \left(-\frac{1}{y_{01}}\right)^{m-1}[-m,1,1;m,0,0]_y = \left(-\frac{1}{z_{10}}\right)^{m-1}[-m,1,1;m,0,0]_z,$$

which may be written in a form even more clearly expressive of the cyclically persistent property, viz.,

$$\left(\frac{1}{x_{10}x_{01}}\right)^{(m-1)/3}[-m,\,1,\,1\,;\,m,\,0,\,0]_x = \left(\frac{1}{y_{10}y_{01}}\right)^{(m-1)/3}[-m,\,1,\,1\,;\,m,\,0,\,0]_y$$
$$= \left(\frac{1}{z_{10}z_{01}}\right)^{(m-1)/3}[-m,\,1,\,1\,;\,m,\,0,\,0]_z. \quad . \quad (112)$$

A first case of this formula occurs in (96). A second is that the operator on the right of (109) is the sum of terms involving z_{10}, z_{01}, and of an operator free from those derivatives which, when affected with the factor $(1/z_{10}z_{01})^{\frac{1}{3}}$, has the persistent property. The fact is interesting in its bearing on the theory of the transformation of ω_1 and ω_2. In fact (109) may be written

$$\omega_1(x, yz) = \omega_2(y, zx) = z_{10}\omega_2(z, xy) + z_{01}\omega_1(z, xy) + [-2, 1, 1; 2, 0, 0]_z.$$

From this and its two correlatives a verification of the formula (112) for the case of $m = 2$ is readily obtained. The first few terms of this cyclically persistent operator are

$$\left(\frac{1}{x_{10}x_{01}}\right)^{\frac{1}{3}}\left\{2\frac{x_{20}{}^2}{2}\frac{d}{dx_{40}} + 2x_{20}x_{11}\frac{d}{dx_{31}} + 2\left(x_{20}x_{02} + \frac{x_{11}{}^2}{2}\right)\frac{d}{dx_{22}} + 2x_{11}x_{02}\frac{d}{dx_{13}} + 2\frac{x_{02}{}^2}{2}\frac{d}{dx_{04}}\right.$$
$$+ 3x_{20}x_{30}\frac{d}{dx_{50}} + 3(x_{20}x_{21} + x_{11}x_{30})\frac{d}{dx_{41}} + 3(x_{20}x_{12} + x_{11}x_{21} + x_{02}x_{30})\frac{d}{dx_{32}}$$
$$+ 3(x_{20}x_{03} + x_{11}x_{12} + x_{02}x_{21})\frac{d}{dx_{23}} + 3(x_{11}x_{03} + x_{02}x_{12})\frac{d}{dx_{14}} + 3x_{02}x_{03}\frac{d}{dx_{05}}$$
$$\left. + \ldots\right\} \quad \ldots \quad (113)$$

With it compare the analogous operator (30) of the binary theory. That, like this, is, of course, one of a whole class of persistent operators.

24. The transformation of the four form annihilators Ω_1, Ω_2, V_1, V_2 of pure cyclicants* will be now considered. They are all operators free from first derivatives, and might as such be treated by the methods of the last article. It seems preferable, however, to take them in connection with the formulæ of transformation

$$\frac{d}{dx_{10}} = -\{0, 0, 1; 2, 0, -1\}_y = -\{0, 1, 0; 1, -1, 1\}_z, \quad \ldots \quad (114)$$

$$\frac{d}{dx_{01}} = -\{0, 0, 1; 1, 1, -1\}_y = -\{0, 1, 0; 2, -1, 0\}_z, \quad \ldots \quad (114\text{A})$$

which are particular cases of (82) and (83), or again of (87) with $m = 0$.

The forms of the four annihilators are

$$\Omega_1(z, xy) = \sum_{m+n \not< 2}\left\{mz_{mn}\frac{d}{dz_{m-1,\,n+1}}\right\}$$
$$= \eta\frac{d}{d\xi}(\zeta - z_{10}\xi - z_{01}\eta)$$
$$= \eta\frac{d\zeta}{d\xi} - z_{10}\eta$$
$$= \{0, 1, 0; 1, -1, 1\}_z - z_{10}\frac{d}{dz_{01}}, \quad \ldots \quad (115)$$

* See my papers in the 'London Math. Soc. Proc.,' vol. 18, pp. 142, &c.; vol. 19, pp. 6, &c., and pp. 337, &c.; vol. 20, pp. 131, &c.

$$\Omega_2(z, xy) = \sum_{m+n \not< 2} \left\{ n z_{mn} \frac{d}{dz_{m+1,\, n-1}} \right\}$$

$$= \xi \frac{d}{d\eta} (\zeta - z_{10}\xi - z_{01}\eta)$$

$$= \{0, 0, 1\,;\, 1, 1, -1\}_z - z_{01} \frac{d}{dz_{10}}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

$$V_1(z, xy) = \sum_{m+n \not< 4} \left\{ \sum_{r+s \not< 2}^{r+s \not> m+n-2} (r z_{rs} z_{m-r,\, n-s}) \frac{d}{dz_{m-1,\, n}} \right\}$$

$$= \tfrac{1}{2} \frac{d}{d\xi} \{(\zeta - z_{10}\xi - z_{01}\eta)^2\}$$

$$= \zeta \frac{d\zeta}{d\xi} - z_{10}\xi \frac{d\zeta}{d\xi} - z_{01}\eta \frac{d\zeta}{d\xi} - z_{10}(\zeta - z_{10}\xi - z_{01}\eta)$$

$$= \{0, 1, 0\,;\, 2, -1, 0\}_z - z_{10}\{0, 1, 0\,;\, 1, 0, 0\}_z - z_{01}\{0, 1, 0\,;\, 1, -1, 1\}_z$$

$$- z_{10} \left\{ \{1, 0, 0\,;\, 1, 0, 0\}_z - z_{10} \frac{d}{dz_{10}} - z_{01} \frac{d}{dz_{01}} \right\}$$

$$= \{0, 1, 0\,;\, 2, -1, 0\}_z - z_{10}\{1, 1, 0\,;\, 1, 0, 0\}_z$$

$$- z_{01}\Omega_1(z, xy) + z_{10}^2 \frac{d}{dz_{10}}, \quad . \quad . \quad . \quad (117)$$

$$V_2(z, xy) = \sum_{m+n \not< 4} \left\{ \sum_{r+s \not< 2}^{r+s \not> m+n-2} (s z_{rs} z_{m-r,\, n-s}) \frac{d}{dz_{m,\, n-1}} \right\}$$

$$= \tfrac{1}{2} \frac{d}{d\eta} \{(\zeta - z_{10}\xi - z_{01}\eta)^2$$

$$= \{0, 0, 1; 2, 0, -1\}_z - z_{01}\{1, 0, 1; 1, 0, 0\}_z - z_{10}\Omega_2(z, xy) + z_{01}^2 \frac{d}{dz_{01}} \cdot \quad (118)$$

Thus (114A) may be written

$$\frac{d}{dx_{01}} = -\,\Omega_2(y, zx) - y_{01}\frac{d}{dy_{10}} = -\,\{0, 1, 0;\; 2, -1, 0\}_z,$$

and the result of one cyclical interchange in (114) may be written

$$-\,\Omega_1(x, yz) - x_{10}\frac{d}{dx_{01}} = \frac{d}{dy_{10}} = -\,\{0, 0, 1;\; 2, 0, -1\}_z;$$

from which two sets of identities, by aid of the facts that

$$x_{10} : x_{01} : -1 = -1 : y_{10} : y_{01} = z_{01} : -1 : z_{10},$$

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it follows at once that

$$\frac{1}{x_{01}}\Omega_1(x, yz) = -\frac{1}{y_{10}}\Omega_2(y, zx)$$
$$= -z_{01}\{0, 1, 0; 2, -1, 0\}_z + z_{10}\{0, 0, 1; 2, 0, -1\}_{z}, . \quad (119)$$

which, and its correlatives, obtained by one and two cyclical transformations effect the transformation of Ω_1 and Ω_2.

Again, we may write (114A)

$$\frac{d}{dx_{01}} = -\Omega_2(y, zx) - y_{01}\frac{d}{dy_{10}}$$
$$= -V_1(z, xy) - z_{01}\Omega_1(z, xy) - z_{10}\{1, 1, 0; 1, 0, 0\}_z + z_{10}{}^2\frac{d}{dz_{10}},$$

and (114) in like manner

$$\frac{d}{dx_{10}} = -V_2(y, zx) - y_{10}\Omega_2(y, zx) - y_{01}\{1, 0, 1; 1, 0, 0\}_y + y_{01}{}^2\frac{d}{dy_{01}}$$
$$= -\Omega_1(z, xy) - z_{10}\frac{d}{dz_{01}}.$$

From these it follows at once that

$$x_{10}\frac{d}{dx_{10}} + x_{01}\frac{d}{dx_{01}} = y_{10}\frac{d}{dy_{10}} + y_{01}\frac{d}{dy_{01}} - \frac{1}{y_{01}}V_2(y, zx) - \{1, 0, 1; 1, 0, 0\}_y$$
$$= z_{10}\frac{d}{dz_{10}} + z_{01}\frac{d}{dz_{01}} - \frac{1}{z_{10}}V_1(z, xy) - \{1, 1, 0; 1, 0, 0\}_z$$

By a cyclical interchange of the variables we have, also,

$$x_{10}\frac{d}{dx_{10}} + x_{01}\frac{d}{dx_{01}} - \frac{1}{x_{10}}V_1(x, yz) - \{1, 1, 0; 1, 0, 0\}_x = y_{10}\frac{d}{dy_{10}} + y_{01}\frac{d}{dy_{01}}$$
$$= z_{10}\frac{d}{dz_{10}} + z_{01}\frac{d}{dz_{01}} - \frac{1}{z_{01}}V_2(z, xy) - \{1, 0, 1; 1, 0, 0\}_z;$$

and by a second cyclical interchange a third set of such equalities is obtained. From the two sets that have been written out, upon subtraction we obtain

$$\frac{1}{x_{10}}V_1(x, yz) + \{1, 1, 0; 1, 0, 0\}_x = -\frac{1}{y_{01}}V_2(y, zx) - \{1, 0, 1; 1, 0, 0\}_y$$
$$= -\frac{1}{z_{10}}V_1(z, xy) + \frac{1}{z_{01}}V_2(z, xy) + \{0, -1, 1; 1, 0, 0\}_z$$

Now, from (91) and its correlative obtained by a cyclical interchange, the second parts of these three equal operators are themselves equal. Consequently

$$\frac{1}{x_{10}} V_1 (x, yz) = -\frac{1}{y_{01}} V_2 (y, zx) = -\frac{1}{z_{10}} V_1 (z, xy) + \frac{1}{z_{01}} V_2 (z, xy), \quad . \quad (120)$$

which, and its correlatives, are the formulæ for the transformation of V_1 and V_2.

It is easy and very instructive to prove (120) directly from the symbolical expressions in (117) and (118) by the method of Art. 19 or 23.

Of other important operators Mr. FORSYTH's Δ_4 and Δ_3 ('Phil. Trans.,' A., vol. 180, p. 74) should have their formulæ of transformation noted. They are the complete operators $\{0, 1, 0; 1, -1, 1\}$ and $\{0, 0, 1; 1, 1, -1\}$ of which Ω_1 and Ω_2 are all but the first terms. Thus their formulæ of transformation are merely (114) and (114A) themselves, *i.e.*, cyclically interchanging the variables once,

$$\Delta_4 (x, yz) = -\frac{d}{dy_{10}} = \{0, 0, 1; 2, 0, -1\}_z, \quad . \quad . \quad . \quad . \quad (121)$$

$$\Delta_3 (x, yz) = \{0, 1, 0; 2, -1, 0\}_y = -\frac{d}{dz_{01}}. \quad . \quad . \quad . \quad . \quad (122)$$